GENERALIZATIONS OF COMPLETELY CONVEX FUNCTIONS

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A function $f(x)$ is said to be completely convex in an interval if it is real-valued and has derivatives of all orders (is of class C^∞) and if $(-1)^n f^{(2n)}(x) \geqq 0$ in that interval. D. V. Widder has shown that a completely convex function coincides in its interval of definition with an entire function of growth not exceeding order one and finite type.[1] One of the present authors has given a new proof of this theorem;[2] the other, a generalization.[3] In this note we give a much wider generalization, which also contains some results of S. Bernstein. This generalization was started by an (unpublished) note of the first named author, where a part of Theorem 2 (stated below) was proved, namely that $f(x)$ is entire and of finite order in case (I).

THEOREM 1. *Let $\{n_k\}$ and $\{q_k\}$ be sequences of positive integers, $\{n_k\}$ strictly increasing. Let $f(x)$ be real-valued and of class C^∞ in $-1 \leqq x \leqq 1$. For $k = 1, 2, \ldots$, let $f^{(n_k)}(x)$ and $f^{(n_k + 2q_k)}(x)$ not change sign in $(-1, 1)$, and let*

$$f^{(n_k)}(x) f^{(n_k + 2q_k)}(x) \leqq 0. \qquad (1)$$

(I) *If $n_k - n_{k-1} = O(1)$ and $q_k = O(1)$, then $f(x)$ coincides in $(-1, 1)$ with an entire function of growth not exceeding order one and finite type.*

(II) *If $n_k - n_{k-1} = O(n_k^\delta)$, $q_k = O(n_k^\delta)$, and $q_1 + q_2 + \ldots + q_k = O(n_k)$, where δ is fixed, $0 < \delta < 1$, then $f(x)$ coincides in $(-1, 1)$ with an entire function of finite order not exceeding $1/(1-\delta)$.*

(III) *If $n_k - n_{k-1} = o(n_k)$, $q_k = o(n_k)$, and $q_1 + q_2 + \ldots + q_k = O(n_k)$, then $f(x)$ coincides in $(-1, 1)$ with an entire function.*

This theorem contains (for $q_k = 1$) certain results of S. Bernstein,[4] who derived them on the more restrictive hypothesis that no derivative of $f(x)$ changes sign in $(-1, 1)$. An interesting special case (where $2q_k = n_{k+1} - n_k$) is the following, a direct generalization of Widder's result.

THEOREM 2. *Let $\{n_k\}$ be a strictly increasing sequence of positive even integers. Let $f(x)$ be real-valued and of class C^∞ in $(-1, 1)$, and let*

$$(-1)^k f^{(n_k)}(x) \geqq 0 \qquad (k = 1, 2, \ldots). \qquad (2)$$

(I) *If $n_k - n_{k-1} = O(1)$, $f(x)$ coincides in $(-1, 1)$ with an entire function of growth not exceeding order one and finite type.*

(II) *If $n_k - n_{k-1} = O(n_k^\delta)$, $0 < \delta < 1$, $f(x)$ coincides in $(-1, 1)$ with an entire function of finite order not exceeding $1/(1-\delta)$.*

(III) *If* $n_k - n_{k-1} = o(n_k)$, $f(x)$ *coincides in* $(-1, 1)$ *with an entire function.*

THEOREM 3. *The results stated in Theorems* 1 *and* 2 *are "best" results in the following sense:*

(I, II) *Corresponding to a given* δ, $0 < \delta < 1$, *we can construct an increasing sequence* $\{n_k\}$ *of even integers and an entire function* $f(x)$ *such that* (2) *is satisfied,* $(n_k - n_{k-1})n_k^{-\delta}$ *tends to a positive limit, and* $f(x)$ *is exactly of order* $1/(1 - \delta)$.

(III) *Corresponding to a given positive* ϵ *we can construct an increasing sequence* $\{n_k\}$ *of even integers and a function* $f(x)$, *analytic in* $(-1, 1)$, *such that* (2) *is satisfied,* $(n_k - n_{k-1})/n_k$ *tends to* ϵ, *but* $f(x)$ *is not entire.*

Our proof of Theorem 1, which we give in outline, depends on the following lemma.

LEMMA. *If, in* $-1 \leqq x \leqq 1$, $g(x)$ *is real-valued, of class* C^{p+2q}, *and satisfies*

$$|g(x)| \leqq M, \quad g^{(p+2q)}(x) \leqq 0,$$

then

$$g^{(p)}(x) \leqq A^{p+2q}(p + 2q)^p M$$

in $(-1, 1)$, A *being an absolute constant (independent of the positive integers* p *and* q *and of the function* $g(x)$).

By Taylor's theorem with remainder,

$$g^{(p)}(x) = g^{(p)}(0) + \frac{xg^{(p+1)}(0)}{1!} + \ldots + \frac{x^{2q-1}g^{(p+2q-1)}(0)}{(2q-1)!} + \frac{x^{2q}g^{(p+2q)}(\xi)}{(2q)!},$$

where $-1 < \xi < 1$ since $-1 \leqq x \leqq 1$. By hypothesis, the remainder is not positive and $g^{(p+2q-1)}(x)$ is monotonic. Therefore,[5] in $-1 + h \leqq x \leqq 1 - h$, where $0 < h < 1$,

$$|g^{(p+2q-1)}(x)| \leqq (p + 2q - 1)!\,\frac{1}{2}\left(\frac{4}{h}\right)^{p+2q-1} M.$$

Hence we deduce, by a theorem of A. Gorny,[6] inequalities for $|g^{(\nu)}(0)|$, $\nu = 1, 2, \ldots, p + 2q - 2$; from these, the lemma follows.

To prove Theorem 1, denote the maximum of $|f^{(n)}(x)|$ in $(-1, 1)$ by M_n. We apply the lemma to $g(x) = \pm f^{n_{k-1}}(x)$, with $p = n_k - n_{k-1}$ and $q = q_k$. We obtain

$$M_{n_k} \leqq A^{n_k - n_{k-1} + 2q_k}(n_k - n_{k-1} + 2q_k)^{n_k - n_{k-1}} M_{n_{k-1}}.$$

Discussion of this recursive inequality gives appropriate upper bounds for $M_{n_1}, M_{n_2}, \ldots$. From these we pass, by the same theorem of Gorny, to an appropriate upper bound for $|f^{(n)}(x)|$ in $-1 + \epsilon \leqq x \leqq 1 - \epsilon$, $n = 0, 1, 2, \ldots$.

[1] Widder, D. V., "Functions Whose Even Derivatives Have a Prescribed Sign," these PROCEEDINGS, **26**, 657–659 (1940).

[2] Boas, R. P., Jr., "A Note on Functions of Exponential Type," forthcoming in *Bull. Am. Math. Soc.*

[3] Pólya, G., "On Functions Whose Derivatives Do Not Vanish in a Given Interval," these PROCEEDINGS, **27**, 216–218 (1941).

[4] Bernstein, S., "On Certain Properties of Regularly Monotonic Functions," [in Russian], *Comm. Soc. Math. Kharkow*, (4) **2**, 1–11 (1928).

[5] Landau, E., "Über einen Satz von Herrn Esclangon," *Math. Annalen*, **102**, 177–188 (1929); S. Bernstein, *Leçons sur les propriétés extrémales* ..., Paris, 1926, p. 10.

[6] Gorny, A., "Contribution à l'Étude des Fonctions Dérivables d'Une Variable Réelle," *Acta Math.*, **71**, 317–358 (1939).

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*NATURAL SYSTEMS: THE STRUCTURE OF ABSTRACT MONOTONE SEQUENCES**

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1. In a previous communication[1] the author introduced the problem: to characterize the possible regular orderings of a natural system. A complete characterization was there given for the simplest non-trivial case, the natural system N_2 (containing exactly 2 primes). In the present paper a statement of the solution of this problem for the general case N_k (k primes, $1 \leqslant k \leqslant \aleph_0$)[2] and also that of the principal lemma are given, without proofs.

2. A natural system (with unit)[3] is an abstract system $(N, \circ)$ with single composition $\circ$ (called simply "product") satisfying (1°)–(7°) (in which, as elsewhere, $\sigma\circ\tau$ is abbreviated by $\sigma\tau$). For all elements σ, τ, φ of the class N

(1°) $\sigma\tau$ is a unique element of N.
(2°) $\sigma(\tau\varphi) = (\sigma\tau)\varphi$.
(3°) $\sigma\tau = \tau\sigma$.
(4°) N contains a unique unit element $\epsilon:(\epsilon\,\sigma = \sigma\,\epsilon = \sigma)$.
(5°) N contains at least one prime (= irreducible) element $\alpha \neq \epsilon:(\alpha = \varphi\psi$ implies $\varphi = \epsilon$ or $\psi = \epsilon)$.
(6°) Each element $\neq \epsilon$ of N can be expressed as the product of (a finite number of) prime elements in exactly one way.
(7°) N is a denumerably infinite class.

The abstractly distinct natural systems may be listed as N_1, N_2, ...,